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"Two Nontrivial Critical Points for Nonsmooth Functionals via Local Linking and Applications"

Dimitrios Kandilakis[1], Nikolaos C. Kourogenis[2] and Nikolaos S. Papageorgiou[3]
[1]*Department of Mathematics, Technical University of Crete, Chania, Crete 73100, Greece*
[2]*Department of Applied Mathematics, University of Crete, Herakleion, Crete 71409, Greece*
[3]*Department of Mathematics, National Technical University, Zografou Campus,Athens 15780, Greece (e-mail: npapg@math.ntua.gr)*

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Abstract. In this paper, we extend to nonsmooth locally Lipschitz functionals the multiplicity result of Brezis–Nirenberg (Communication Pure Applied Mathematics and 44 (1991)) based on a local linking condition. Our approach is based on the nonsmooth critical point theory for locally Lipschitz functions which uses the Clarke subdifferential. We present two applications. This first concerns periodic systems driven by the ordinary vector p-Laplacian. The second concerns elliptic equations at resonance driven by the partial p-Laplacian with Dirichlet boundary condition. In both cases the potential function is nonsmooth, locally Lipschitz.

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1. Introduction

The notion of local linking was first introduced by Liu and Li [23]. Soon thereafter Brezis and Nirenberg [3] relaxed the assumptions for local linking and proved a theorem on the existence of two nontrivial critical points for a C^1-functional satisfying the Palais–Smale condition. Their approach used the Ekeland variational principle and a deformation theorem.

The purpose of this paper is to present a generalization of the multiplicity result of Brezis and Nirenberg. In this generalization we do not require the energy functional to be smooth, it is only locally Lipschitz. Also instead of the usual Palais–Smale condition, we employ the more general Cerami condition (its nonsmooth version). However, this is no real improvement since as it was shown by Kourogenis–Papageorgiou [9] for functionals bounded below, the two conditions are equivalent. In the second half of the paper we present applications of the abstract multiplicity result to nonlinear periodic systems and to nonlinear elliptic equations with nonsmooth potential, known as "hemivariational inequalities". Such inequality problems arise in mechanics when one wants to consider more realistic models with nonsmooth and nonconvex energy functionals. For such applications we refer to the book of Naniewicz and Panagiotopoulos [28]. For the mathematical aspects of the theory of hemivariational inequalities, we refer to the works of Degiovanni et al. [8], Gasinski and Papageorgiou [12,13], Goeleven et al. [14], Kyritsi and Papageorgiou [21], Motreanu and Radulescu [27], Radulescu and Panagiotopoulos [29] and the references therein. Also problems with discontinuities can be studied within the mathematical framework of hemivariational inequalities. We refer to the works of Chang [5], Costa and Goncalves [7], and Hu et al. [15].

Our approach is based on the nonsmooth critical point theory for locally Lipschitz functionals, as this was originally formulated by Chang [5] and extended recently by Kourogenis and Papageorgiou [18] and Kyritsi and Papageorgiou [20]. In the next section for the convenience of the reader we recall some basic definitions and facts from this theory.

2. Mathematical Preliminaries

The nonsmooth critical point theory for locally Lipschitz functionals is based on the subdifferential theory of Clarke [6].

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). A function $\phi\colon X \to \mathbb{R}$ is said to be locally Lipschitz, if for every bounded set $B \subseteq X$ there exists $k_B > 0$ such that $|\varphi(x) - \varphi(y)| \leqslant k_B \|x - y\|$ for all $x, y \in B$. This is a slightly more restrictive version of local Lipschitzness than the one used in the literature (see, for example [6,19]), where φ is locally Lipschitz, if for every $u \in X$, there exists a neighborhood U of u and a constant $k_U > 0$ such that $|\varphi(x) - \varphi(y)| \leqslant k_U \|x - y\|$ for all $x, y \in U$. However, for the applications on nonsmooth boundary value problems that we have in mind this more restrictive definition is sufficient and always satisfied. Note that if X is finite dimensional, then the two definitions are equivalent. From convex analysis we know that a convex and lower semicontinuous function $g\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ for which $dom\, g = \{x \in X : g(x) < +\infty\} \neq \emptyset$, it is locally Lipschitz in $int\, dom\, g$. In analogy with the directional derivative of a convex function, we define the generalized directional derivative of a locally Lipschitz function ϕ at $x \in X$ in the direction $h \in X$, by

$$\phi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\phi(x' + \lambda h) - \phi(x')}{\lambda}.$$

It is easy to see that the function $h \to \phi^0(x; h)$ is sublinear and continuous on X. So by the Hahn–Banach theorem it is the support function of a nonempty, convex, w^*-compact set

$$\partial\phi(x) = \{x^* \in X^* : (x^*, h) \leqslant \phi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\phi(x)$ is known as the generalized (or Clarke) subdifferential of ϕ. If $\phi, \psi : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$ and for every $\lambda \in \mathbb{R}$ $\partial(\lambda\phi)(x) = \lambda\partial\phi(x)$. Moreover, if ϕ is also convex, then this subdifferential coincides with the subdifferential in the sense of convex analysis (i.e. $\partial\phi(x) = \{x^* \in X^* : (x^*, y - x) \leqslant \phi(y) - \phi(x) \text{ for all } y \in X\} = \{x^* \in X^* : (x^*, h) \leqslant \phi'(x; h) \text{ for all } h \in X\}$, with

$$\phi'(x; h) = \lim_{\lambda \downarrow 0} \frac{\phi(x + \lambda h) - \phi(x)}{\lambda} = \inf_{\lambda > 0} \frac{\phi(x + \lambda h) - \phi(x)}{\lambda},$$

the directional derivative of the convex function ϕ). If $\phi \in C^1(X)$, then $\partial\phi(x) = \{\phi'(x)\}$.

Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. A point $x \in X$ is said to be a critical point of ϕ if $0 \in \partial\phi(x)$. If $x \in X$ is a critical point of ϕ, then $c = \phi(x)$ is a critical value of ϕ. It is easy to see that, if $x \in X$ is a local extremum of ϕ, then $0 \in \partial\phi(x)$. Moreover, the multifunction $x \to \partial\phi(x)$ is upper semicontinuous from X into X^* equipped with the weak* topology, i.e. for any $U \subseteq X^*$ w^*-open, the set $\{x \in X : \partial\phi(x) \subseteq U\}$ is open in X (see [16]). For more details we refer to Clarke [6].

The critical point theory for smooth functions uses a compactness condition known as the "Palais–Smale condition" (PS). In the present nonsmooth setting this condition takes the following form:

> *"The locally Lipschitz function ϕ: $X \to \mathbb{R}$ satisfies the "nonsmooth PS condition" if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\phi(x_n)\}_{n\geqslant 1}$ is bounded and $m(x_n) = \min[\|x^*\| : x^* \in \partial\phi(x_n)] \to 0$ as $n \to \infty$, has a strongly convergent subsequence".*

If $\phi \in C^1(X)$, then as we already mentioned $\partial\phi(x) = \{\phi'(x)\}$ and so the above definition of the PS condition coincides with the classical (smooth) one (see [30]).

In the context of the smooth theory, Cerami [4] introduced a weaker compactness condition which in our nonsmooth setting has the following form:

> *"The locally Lipschitz function $\phi : X \to \mathbb{R}$ satisfies the "nonsmooth Cerami condition" (nonsmooth C-condition for short), if for any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that the sequence $\{\phi(x_n)\}_{n\geqslant 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence".*

This weaker condition suffices to obtain a deformation theorem and through it derive minimax theorems locating critical points. This was done in the smooth case by Bartolo et al. [1] and in the nonsmooth case by Kourogenis and Papageorgiou [18].

In our analysis we shall need a recent generalization of the Ekeland variational principle due to Zhong [33]. For easy reference, we recall the result.

THEOREM 1. *If* (V, d_V) *is a complete metric space,* $\phi : V \to \overline{\mathbb{R}}$ *is lower semicontinuous, bounded below and not identically* $+\infty$ *and* $v_0 \in V$ *is fixed, then for every* $\varepsilon > 0$*, every* $y \in V$ *such that* $\phi(y) \leqslant \inf_V \phi + \varepsilon$ *and every* $\lambda > 0$*, there exists* $v \in V$ *such that*

(a) $\phi(v) \leqslant \phi(y)$,
(b) $d_V(v, v_0) \leqslant r_0 + \overline{r}$ *and*
(c) $\phi(v) - \frac{\varepsilon}{\lambda(1+d_V(v_0,v))} d_V(v, z) \leqslant \phi(z)$ *for all* $z \in V$

with $r_0 = d_V(v_0, y)$ *and* $\overline{r} > 0$ *such that* $\int_{r_0}^{r_0+\overline{r}} 1/(1+r)\,\mathrm{d}r \geqslant \lambda$.

Finally in the applications to nonlinear elliptic equations, we use the principal eigenvalue of the negative p-Laplacian with Dirichlet boundary condition, i.e. of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$. So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^{1,a}$-boundary Γ $(0 < a < 1)$. We consider the following eigenvalue problem:

$$\left\{ \begin{array}{l} -div(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) \text{ a.e. on } Z \\ x|_\Gamma = 0. \end{array} \right\}.$$

The least $\lambda \in \mathbb{R}$ for which this problem has a nontrivial solution is called the first (or principal) eigenvalue of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$ and is denoted by λ_1. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenspace is one-dimensional). Moreover, we have a variational characterization of λ_1 via the Rayleigh quotient, i.e.

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \neq 0\right].$$

This minimum is realized at the normalized principal eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that u_1 does not change sign on Z. In fact we can show that $u_1 \neq 0$ a.e. on Z and from nonlinear regularity theory (see Lieberman [24]) we can have that $u_1 \in C^1(\overline{Z})$. For details we refer to [25].

3. Multiple Critical Points via Local Linking

In this section, we prove an abstract multiplicity result under a local linking condition, extending this way the result of Brezis and Nirenberg [3].

So let X be a Banach space and $\phi: X \to \mathbb{R}$ a locally Lipschitz function such that $\phi(0)=0$.

PROPOSITION 2. *If ϕ is bounded below and satisfies the nonsmooth C-condition, then ϕ attains its infimum on X.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\phi(x_n) \downarrow \inf_X \phi$. Using Theorem 1 with $V=X$, $v_0=0$, $\varepsilon=(1/n^2)$ and $\lambda=\sqrt{\varepsilon}$, we obtain a sequence $\{y_n\}_{n\geqslant 1} \subseteq X$ such that for all $n \geqslant 1$ we have

$$\phi(y_n) \leqslant \phi(x_n) \text{ (hence } \phi(y_n) \downarrow \inf_X \phi), \quad \|y_n\| \leqslant \|x_n\| + \bar{r}$$

$$\text{and } \phi(v) \geqslant \phi(y_n) - \frac{1}{n(1+\|y_n\|)}\|v-y_n\| \quad \text{for all } v \in X,$$

where $\bar{r}>0$ is such that $\int_{\|x_n\|}^{\|x_n\|+\bar{r}} 1/(1+r)\,\mathrm{d}r \geqslant (1/n)$. For every $v \in X$ and every $t \in (0,1)$, we have

$$-\frac{t\|v-y_n\|}{n(1+\|y_n\|)} \leqslant \phi(y_n + t(v-y_n)) - \phi(y_n)$$

$$\Rightarrow -\|v-y_n\| \leqslant n(1+\|y_n\|)\frac{\phi(y_n+t(v-y_n))-\phi(y_n)}{t}.$$

Set $v=y_n+h$, $h \in X$ and $\eta_n(h)=\phi^0(y_n;h)$. We know that $\eta_n(\cdot)$ is sublinear, continuous and so $\eta_n(0)=0$. We have

$$-\|h\| \leqslant n(1+\|y_n\|)\eta_n(h) \quad \text{for all } h \in X.$$

By virtue of Lemma 1.3 of Szulkin [31], we obtain $u_n^* \in X^*$, $\|u_n^*\| \leqslant 1$ such that

$$(u_n^*, h) \leqslant n(1+\|y_n\|)\eta_n(h) \quad \text{for all } h \in X.$$

Set $v_n^* = 1/(n(1+\|y_n\|))u_n^*$. It follows that $v_n^* \in \partial\phi(y_n), n \geqslant 1$, so

$$(1+\|y_n\|)m(y_n) \leqslant \frac{1}{n}\|u_n^*\| \leqslant \frac{1}{n} \to 0.$$

Since by hypothesis ϕ satisfies the nonsmooth C-condition, by passing to a subsequence if necessary, we may assume that $y_n \to y_0$ in X. Evidently $\phi(y_n) \to \phi(y_0)$ and so $\phi(y_0) = \inf_X \phi$. □

Let $y_0 \in X$ be the global minimizer of ϕ on X established by Proposition 2. We assume that $\inf_X \phi < 0 = \phi(0)$, hence $y_0 \neq 0$. Also let U be a neighborhood of y_0, $B_\delta = \{x \in X : \|x\| < \delta\}$, $U \cap B_\delta = \emptyset$ and for $c \in \mathbb{R}$ let $\phi^c = \{x \in X : \phi(x) \leqslant c\}$. Evidently if $c \geqslant \inf_X \phi$, then $\phi^c \neq \emptyset$.

PROPOSITION 3. *If ϕ is bounded below and satisfies the nonsmooth C-condition, $c \geqslant \inf_X \phi$ and $\{y_0, 0\}$ are the only critical points of ϕ, then there exists $\gamma > 0$ such that $(1 + \|x\|)m(x) \geqslant \gamma$ for all $x \in \phi^c \setminus (U \cup B_\delta)$.*

Proof. Suppose that the conclusion of the proposition is not true. Then we can find $x_n \in \phi^c \setminus (U \cup B_\delta)$ such that $(1 + \|x_n\|)m(x_n) \to 0$. Since ϕ satisfies the nonsmooth C-condition we may assume that $x_n \to x$ in X. From Chang [5] we know that $m(\cdot)$ is lower semicontinuous. So $m(x) \leqslant \liminf_{n\to\infty} m(x_n) = 0$, hence $m(x) = 0$. Therefore, $0 \in \partial\phi(x)$ and since by hypothesis y_0 and 0 are the only critical points of ϕ, it follows that $x = y_0$ or $x = 0$. But $x \in \phi^c \setminus (U \cup B_\delta)$ since the latter set is closed, we have a contradiction. $\square$

PROPOSITION 4. *If the hypotheses of Proposition 3 hold and $c > 0$, then there exists $v\colon \phi^c \setminus (U \cup B_\delta) \to X$ a locally Lipschitz map such that*

$$\|v(x)\| \leqslant 1 + \|x\| \;\; \textit{and for all } x^* \in \partial\phi(x) \;\; (x^*, v(x)) \geqslant \frac{\gamma}{2}$$

with $\gamma > 0$ as in Proposition 3.

Proof. Let $D = \phi^c \setminus (U \cup B_\delta)$ and let $B(0, m(x)) = \{z^* \in X^* : \|z^*\| < m(x)\}$. Evidently we have $B(0, m(x)) \cap \partial\phi(x) = \emptyset$. Since both sets are convex and $B(0, m(x))$ is open, we can apply the weak separation theorem and find $u(x) \in X$ with $\|u(x)\| = 1$ such that

$$(z^*, u(x)) \leqslant (x^*, u(x)) \qquad \text{for all } z^* \in B(0, m(x)) \text{ and all } x^* \in \partial\phi(x).$$

Note that $\sup[(z^*, u(x)) : z^* \in B(0, m(x))] = m(x)$. So using Proposition 3, we have

$$\frac{\gamma}{2(1 + \|x\|)} < m(x) \leqslant (x^*, u(x)) \text{ for all } x^* \in \partial\phi(x) \text{ and all } x \in D. \qquad (1)$$

Recall (see Section 2) that the multifunction $x \to \partial\phi(x)$ is usc from X into X^*_w (i.e. X^* with the weak topology). So $x \to (1 + \|x\|)\partial\phi(x)$ is usc from X into X^*_w. Let $x \in D$ and let $V = \{y^* \in X^* : (\gamma/2) < (y^*, u(x))\}$. Evidently V is a weakly open subset of X^* and from (1) we see that $(1 + \|x\|)\partial\phi(x) \subseteq V$. Then we can find $\theta(x) > 0$ such that for all $y \in B(x, \theta(x)) \cap D$ $(B(x, \theta(x)) = \{y \in X : \|x - y\| < \theta(x)\})$, we have

$$(1+\|y\|)\partial\phi(y)\subseteq V$$

$$\Rightarrow \frac{\gamma}{2(1+\|y\|)} < (y^*, u(x)) \text{ for all } y^*\in\partial\phi(y). \tag{2}$$

The collection $\{B(x,\theta(x))\}_{x\in D}$ is an open cover of D. By paracompactness we can find a locally finite refinement $\{U_i\}_{i\in I}$ and a locally Lipschitz partition of unity $\{\xi_i\}_{i\in I}$ subordinate to it. For each $i\in I$ we can find $x_i\in D$ such that $U_i\subseteq B(x_i,\theta(x_i))$. To this $x_i\in D$ corresponds the element $u_i=u(x_i)\in X$ with $\|u_i\|=1$, for which (1) holds with $x=x_i$. Now, let $v: D\to X$ be defined by

$$v(x)=(1+\|x\|)\sum_{i\in I}\xi_i(x)u_i.$$

Evidently this map is well defined, locally Lipschitz and

$$\|v(x)\|\leqslant 1+\|x\|.$$

Moreover, from (2) we see that for every $x^*\in\partial\phi(x)$ we have

$$(x^*, v(x))=\sum_{i\in I}\xi_i(x)(1+\|x\|)(x^*,u_i)\geqslant\frac{\gamma}{2}\sum_{i\in I}\xi_i(x)=\frac{\gamma}{2}. \qquad \square$$

We continue with the hypotheses of Proposition 3 in effect. Consider the open set $U=\{x\in X:\phi(x)<\phi(y_0)+\delta\}$, $\delta>0$. Also let $\xi>0$ small so that for all $x\in\overline{B}(y_0,\xi)=\{x\in X:\|x-y_0\|\leqslant\xi\}$, we have $\phi(x)\leqslant 0$.

We can choose $\delta>0$ small so that $\phi(y_0)+\delta<0$ and

$$U\subseteq\{x\in X:\|x-y_0\|<\xi\}=B(y_0,\xi).$$

Indeed, if no such $\delta>0$ exists, we can find $\{x_n\}_{n\geqslant 1}\subseteq X$ such that $\phi(x_n)\downarrow\inf_X\phi$ and $\|x_n-y_0\|\geqslant\xi$. Using the Ekeland variational principle (see, for example [16], p.520) we can find $\{y_n\}_{n\geqslant 1}\subseteq X$ such that $\varphi(y_n)\leqslant\varphi(x_n)$ (hence $\varphi(y_n)\downarrow\inf_X\varphi$), $\mathrm{d}(x_n,y_n)\to 0$ and $m(y_n)\to 0$ as $n\to\infty$. Note that since φ is bounded below and satisfies the nonsmooth C-condition, it satisfies the nonsmooth PS-condition (see [19]). Hence we may assume that $y_n\to y$ in x. Then $x_n\to y$ in X and we have $\|y-y_0\|\geqslant\xi$, i.e. $y\neq y_0$ and also $y\neq 0$ (since $\varphi(y)=\inf_X\varphi<0=\varphi(0)$), a contradiction to the hypothesis that $\{y_0,0\}$ are the only critical points of φ.

Without any loss of generality assume that $1<\|y_0\|$. Also assume that

$$X=Y\oplus V$$

with $dimY < +\infty$. Choose $\delta > 0$ such that $int\,[\phi^c \setminus (U \cup B_\delta)] \neq \emptyset$ and $z \in int\,[\phi^c \setminus (U \cup B_\delta)]$ with $\phi(z) \leqslant -k\delta < 0 = \phi(0)$, where $k > 0$ is the Lipschitz constant of ϕ on B_δ. Note that since $c > 0$, if we choose $\delta > 0$ small enough we will satisfy the above requirements. Now let $v : D = \phi^c \setminus (U \cup B_\delta) \to X$ be the locally Lipschitz map obtained in Proposition 4. We consider the following Cauchy problem:

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\dfrac{v(\eta(t))}{\|v(\eta(t))\|^2} \text{ on } \mathbb{R}_+, \\ \\ \eta(0) = z. \end{array} \right\}. \tag{3}$$

It is well-known that (3) has a unique flow.

PROPOSITION 5. *If the hypotheses of Proposition* 3 *hold and* $c > 0$, $z \in X$ *and* $\delta > 0$ *are chosen as above, then there exists a finite time* $\tau(z) < +\infty$ *such that the flow of* (3) *exists on* $[0, \tau(z)]$ *and* $\phi(\eta(\tau(z))) = \phi(y_0) + \delta$.

Proof. Since $z \in int\,[\phi^c \setminus (U \cup B_\delta)] = int\, D$, the solution of (3) exists on a maximal open interval $[0, \tau(z))$. Remark that the function $t \to \phi(\eta(t))$ is locally Lipschitz, thus differentiable almost everywhere on $(0, \tau(z))$. From Chang [5] (p. 106), we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(\eta(t)) \leqslant \max[(x^*, \eta'(t)) : x^* \in \partial\phi(\eta(t))]$$

$$= \max\left[\left(x^*, -\frac{v(\eta(t))}{\|v(\eta(t))\|^2}\right) : x^* \in \partial\phi(\eta(t))\right] \leqslant -\gamma_0 < 0$$

a.e. on $[0, \tau(z))$, with $\gamma_0 > 0$.

The last inequality is a consequence of Proposition 4. From this we infer that the function $t \to \phi(\eta(t))$ is strictly decreasing on $[0, \tau(z))$ and $\tau(z) \leqslant (1/\gamma_0)(\phi(z) - \inf\phi) < +\infty$. We have

$$\phi(y_0) + \delta \leqslant \phi(\eta(t)) < \phi(z) \quad \text{for all } t \in (0, \tau(z)).$$

From Proposition 3 we know that

$$(x^*, v(\eta(t))) \geqslant \frac{\gamma}{2} \quad \text{for all } t \in [0, \tau(z)) \quad \text{and all } x^* \in \partial\phi(\eta(t)),$$

$$\Rightarrow \|x^*\|\,\|v(\eta(t))\| \geqslant \frac{\gamma}{2} \quad \text{for all } t \in [0, \tau(z)) \quad \text{and all } x^* \in \partial\phi(\eta(t)).$$

Note that for all $t \in [0, \tau(z))$, $\eta(t) \in \phi^c \setminus (U \cup B_\delta) = D$. Because ϕ is locally Lipschitz, bounded below and satisfies the nonsmooth C-condition, it is coercive (see [19]). So D is bounded. Because the subdifferential multifunction $x \to \partial\phi(x)$ is bounded (see [6], p. 27 and recall that according to our definition φ is bounded on bounded sets), it follows that the set $\partial\phi(D) = \bigcup_{x\in D} \partial\phi(x) \subseteq X^*$ is bounded. Therefore, we can find $\gamma_1 > 0$ such that for all $t \in [0, \tau(z))$ and all $x^* \in \partial\phi(\eta(t))$, we have $\|x^*\| \leqslant \gamma_1$. So finally we can say that

$$\|v(\eta(t))\| \geqslant \frac{\gamma}{2\gamma_1} \quad \text{for all } t \in [0, \tau(z)).$$

Therefore, $\int_0^{\tau(z)} \eta'(t)\,\mathrm{d}t$ exists and this means that $\lim_{t\to\tau(z)} \eta(t) = \eta(\tau(z))$ exists. Evidently we must have $\eta(\tau(z)) \in bd(\phi^c \cap (U \cup B_\delta)^c)$. We know that

$$bd(\phi^c \cap (U \cup B_\delta)^c) \subseteq bd\phi^c \cup bd(U \cup B_\delta)^c.$$

Remark that $\phi(\eta(\tau(z))) < \phi(z) < c$ and so $\eta(\tau(z)) \notin bd\phi^c$. Therefore, we must have that

$$\eta(\tau(z)) \in bd(U \cup B_\delta)^c = bd(U^c \cap B_\delta^c) \subseteq bdU^c \cup bdB_\delta^c.$$

If $\eta(\tau(z)) \in bdB_\delta^c = bdB_\delta$, then $\|\eta(\tau(z))\| = \delta$. Since $\phi(0) = 0$ and $\phi|_{\overline{B}_\delta}$ is Lipschitz continuous with constant k, we have $|\phi(\eta(\tau(z)))| \leqslant k\delta$, hence $-k\delta \leqslant \phi(\eta(\tau(z))) \leqslant k\delta$. But recall that $\phi(\eta(\tau(z))) < \phi(z) \leqslant -k\delta$ (from the choice of z), a contradiction. Therefore, $\eta(\tau(z)) \notin bdB_\delta^c$ and so we must have that $\eta(\tau(z)) \in bdU^c = bdU$. We conclude that $\phi(\eta(\tau(z))) = \phi(y_0) + \delta$. □

In the proof of the main multiplicity result, we shall need two more auxiliary results which extend Lemma 1 and Proposition 4 of Brezis and Nirenberg [3].

PROPOSITION 6. *If* $R = \{x \in X : 0 \leqslant a \leqslant \|x\| \leqslant b\}$, $\phi : R \to R$ *is locally Lipschitz, satisfies the nonsmooth C-condition on every closed subset of* $\operatorname{int} R$ *and does not have any critical points in* $\operatorname{int} R$, *then the function* $\xi(r) = \inf[\phi(x) : \|x\| = r]$ *satisfies: if* $a < r_1 < r < r_2 < b$, *then we have*

$$\xi(r) > \min[\xi(r_1), \xi(r_2)].$$

Proof. First note that since R is bounded, on it the nonsmooth C-condition and the nonsmooth PS-condition are equivalent. Now suppose that the conclusion of the Proposition is not true. We can find $r_1 < r < r_2$ such that $\xi(r) \leqslant \min[\xi(r_1), \xi(r_2)]$. Let $\{x_n\}_{n\geqslant 1} \subseteq R$ such that $\|x_n\| = r$ and $\phi(x_n) < \xi(r)$

$+(1/n^2)$. From the Ekeland variational principle (see [31], p. 94) applied on the ring $R_1 = \{x \in X : r_1 \leqslant \|x\| \leqslant r_2\}$, we can find $y_n \in R_1$ such that

$$\phi(v) \geqslant \phi(y_n) - \frac{1}{n}\|v - y_n\| \quad \text{for all } v \in R_1$$

$$\text{and } \phi(y_n) \leqslant \phi(x_n) - \frac{1}{n}\|x_n - y_n\|, \quad n \geqslant 1.$$

We claim that $y_n \notin bd R_1$. Indeed, if $y_n \in bd R_1$, say $\|y_n\| = r_1$, we have

$$\xi(r_1) \leqslant \phi(x_n) - \frac{1}{n}(r - r_1) \leqslant \xi(r) + \frac{1}{n^2} - \frac{1}{n}(r - r_1),$$

which for $n \geqslant 1$ large leads to a contradiction of the hypothesis that $\xi(r) \leqslant \min[\xi(r_1), \xi(r_2)]$. So we have that $y_n \notin bd R_1$ (at least for large $n \geqslant 1$). Therefore for every $n \geqslant 1$, if $h \in X$ and $t \in (0, 1)$ is small, we have $v = y_n + th \in R_1$ and so

$$\phi(y_n + th) - \phi(y_n) \geqslant -\frac{t}{n}\|h\|$$

$$\Rightarrow -\|h\| \leqslant n\frac{\phi(y_n + th) - \phi(y_n)}{t}$$

$$\Rightarrow -\|h\| \leqslant n\phi^0(y_n; h).$$

As before via Lemma 1.3 of Szulkin [31], we obtain $u_n^* \in X^*$ with $\|u_n^*\| \leqslant 1$ such that

$$(u_n^*, h) \leqslant n\phi^0(y_n; h) \quad \text{for all } h \in X$$

$$\Rightarrow \frac{1}{n}u_n^* \in \partial\phi(y_n)$$

$$\Rightarrow m(y_n) \leqslant \frac{1}{n} \to 0.$$

So by passing to a subsequence if necessary, we may assume that $y_n \to y \in R_1$ and $m(y) = 0$, i.e. $0 \in \partial\phi(y)$, a contradiction to the hypothesis that ϕ has no critical points in $int\, R$. $\square$

PROPOSITION 7. *If* $\phi : \overline{B}_R \to \mathbb{R}$ *is locally Lipschitz, satisfies the non-smooth C-condition on every closed subset of* B_R, $\phi(0) = 0$, $\phi(x) > 0$ *for all* $0 < \|x\| < R$ *and has no critical points on* B_R *except* 0, *then there exists* $r_0 \in (0, R]$ *such that* $\xi(r)$ *is strictly increasing on* $[0, r_0)$ *and strictly decreasing on* $[r_0, R)$.

Proof. We know that $\xi(\cdot)$ is upper semicontinuous on $[0, R']$, $R' < R$ (see, for example, [16], p. 82). So we can find $r_0 \in [0, R']$ such that $\xi(r_0) = \max[\xi(r) : 0 \leqslant r \leqslant R']$. Then the result of this Proposition follows from Proposition 6 by letting $R' \to R$. □

Now we are ready for the multiplicity result under local linking.

THEOREM 8. *If X is a reflexive Banach space, $X = Y \oplus V$ with* $\dim Y < +\infty$, *$\phi: X \to \mathbb{R}$ is a locally Lipschitz function which is bounded below, satisfies the nonsmooth C-condition, $\phi(0) = 0$,* $\inf_X \phi < 0$ *and there exists $r > 0$ such that*

$$\left.\begin{aligned} &\phi(x) \leqslant 0 \quad \textit{for } x \in Y,\ \|x\| \leqslant r, \\ &\phi(x) \geqslant 0 \quad \textit{for } x \in V,\ \|x\| \leqslant r, \end{aligned}\right. \quad \textit{local linking},$$

then ϕ has at least two nontrivial critical points.

Proof. We follow the ideas in the proof of Theorem 4 of Brezis and Nirenberg [3]

From Proposition 2 we know that there exists a minimizer $y_0 \neq 0$ of ϕ. Suppose that y_0 and 0 are the only critical points of ϕ. We will derive a contradiction, which shows that there must be at least one more critical point z_0 different from y_0 and 0. This way we will have at least two nontrivial critical points for ϕ.

Case i: $dim\, Y > 0$ and $dim\, V > 0$.

Without any loss of generality, we may assume that $r = 1 < \|y_0\|$. Let $e \in V$ such that $\|e\| = 1$. We introduce the set

$$E = \{x \in X : x = \lambda e + y,\ \ y \in Y, \lambda \geqslant 0, \|x\| \leqslant 1\}.$$

Given $x \in bdE$, $x \neq e$, we have $\|x\| \leqslant 1$ and we can write it in a unique way as

$$x = \lambda e + \mu y$$

with $0 \leqslant \lambda \leqslant 1$, $y \in Y$ with $\|y\| = 1$ and $0 < \mu \leqslant 1$. Evidently by choosing $c > 0$ large and $\delta > 0$ small we can quarantee that if $y \in Y$, $\|y\| = 1$, then we have $y \in int[\phi^c \setminus (U \cup B_\delta)] = int D$. So we can define the map $p^* : bdE \to X$ by

$$p^*(y) = y \quad \text{if } y \in Y \quad \text{with } \|y\| \leqslant 1, \ \ p^*(e) = y_0$$
$$\text{and } p^*(\lambda e + \mu y) = \begin{cases} \eta(2\lambda\tau(y)) & \text{if } \lambda \in [0, 1/2], \\ (2\lambda - 1)y_0 + (2 - 2\lambda)\eta(\tau(y)) & \text{if } \lambda \in (1/2, 1]. \end{cases}$$

Here η and $\tau(y)$ are as in Proposition 5. Clearly p^* is continuous and for all $x \in bdE$ we have $\phi(p^*(x)) \leqslant 0$. Indeed, if $x = y \in Y$ with $\|y\| \leqslant 1 = r$, then from the local linking hypothesis, we have that $\phi(p^*(x))=\phi(y)\leqslant 0$. If $x=e$, then $\phi(p^*(x))=\phi(p^*(e))=\phi(y_0)=\inf_X \phi<0$, by hypothesis. If $x=\lambda e+\mu y$ with $\lambda\in[0,1/2]$, then $\phi(p^*(\lambda e+\mu y))=\phi(\eta(2\lambda\tau(y)))<\phi(\eta(0))$ $=\phi(y)\leqslant 0$ since $y\in Y$ with $\|y\|=1=r$ (local linking hypothesis). Finally, if $x=\lambda e+\mu y$ with $\lambda\in(1/2,1]$, then $p^*(x)=(2\lambda-1)y_0+(2-2\lambda)\eta(\tau(y))$ and as λ moves from $(1/2)$ to 1, then $p^*(x)$ covers the segment $\langle\eta(\tau(y)),y_0\rangle$ $=\{x\in X:(1-\theta)\eta(\tau(y))+\theta y_0,\ 0\leqslant\theta\leqslant 1\}$. So $\|p^*(x)-y_0\|=(2-2\lambda)\|\eta(\tau(y))$ $-y_0\|\leqslant\xi$, hence $\phi(p^*(x))\leqslant 0$.

Note that we can find $0<\gamma_2\leqslant 1$ such that $\|p^*(x)\|\geqslant\gamma_2$ for all $x\in E$ with $\|x\|=1$. We fix $0<\rho<\gamma_2$. From Lemma 3 of Brezis and Nirenberg [3] we know that the sets $p^*(bdE)$ and $S=\{v\in V:\|v\|=\rho\}$ link (i.e. for any continuous extension p of p^* on all of E, we have $p(E)\cap S\neq\emptyset$, see [30], p. 116). Let $\Gamma=\{p\in C(E,X):p|_{bdE}=p^*\}$ and set $c_0=\inf_{p\in\Gamma}\sup_{x\in E}\phi(p(x))$. Note that $c_0\geqslant 0$. Also from Theorem 5 of Kourogenis and Papageorgiou [18], we have that c_0 is a critical value of ϕ. If $c_0>0$, then the corresponding critical point is the second nontrivial critical point of ϕ. If $c_0=0$, then again from Theorem 5 of Kourogenis and Papageorgiou [18], we can produce a critical point of ϕ located on S with critical value c_0. Therefore, we obtain a third critical point distinct from y_0 and 0, a contradiction.

Case ii: $dim\,Y=0$.

If y_0 is the only nonzero critical point of ϕ, then by the local linking hypothesis we can find $\rho_1>0$ such that $\phi(x)>0$ for all $x\neq 0$, $\|x\|<\rho_1$. So by Proposition 7, we can find $\rho_2>0$ small so that for all $\|x\|=\rho_2$ we have $\phi(x)\geqslant\gamma_3>0$. Since $\phi(0)=0>\phi(y_0)$, we can apply the nonsmooth Mountain Pass Theorem of Kourogenis and Papageorgiou [18] to obtain a second nontrivial critical point of ϕ distinct from y_0 and 0, contradicting our initial hypothesis.

Case iii: $dim\,V=0$ (in fact for this case we can allow $dim\,Y=+\infty$).

From Proposition 6 we know that we can find $\rho_3>0$ small so that $\phi(x)\leqslant-\gamma_4<0$ for all $\|x\|=\rho_3$. Also recall that ϕ is coercive (see [19]). So we can apply the nonsmooth Mountain Pass Theorem of Kourogenis and Papageorgiou [18] on the functional $-\phi$ and for paths joining 0 and u with $\phi(u)>0$ and $y_0\notin<0,u>=\{\lambda u,\ \lambda\in[0,1]\}$. So invoking this theorem (see [18], Theorem 6) we obtain a critical point $z_0\in X$ with critical value $\phi(z_0)>\phi(u)>0=\phi(0)>\phi(y_0)$. So $z_0\neq 0$, $z_0\neq y_0$. Therefore, again we have a third critical point z_0 distinct from y_0 and 0, contradicting once more our initial hypothesis. □

4. Applications

In this section, we present applications of Theorem 8 to nonlinear periodic and elliptic problems with nonsmooth potential.We start with periodic systems. So we consider the following problem:

$$\left\{ \begin{array}{l} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in \partial j(t,x(t)) \text{ a.e. on } T=[0,b] \\ \\ x(0)=x(b),\ x'(0)=x'(b),\ 2\leqslant p<\infty \end{array} \right\}. \tag{4}$$

Here $\partial j(t,x)$ stands for the generalized subdifferential of the locally Lipschitz map $j(t,\cdot)$.

In the past, multiplicity results were obtained for scalar (i.e. $N=1$), semilinear (i.e. $p=2$) problems with smooth potential (see, for example [10,11]). Recently Tang [32] considered semilinear systems (i.e. $N>1$) with smooth potential and proved multiplicity results. The corresponding study for quasilinear problems involving the ordinary p-Laplacian is lagging behind. To our knowledge the only work in this direction is that of Del Pino et al. [9], where the problem under consideration is scalar, the nonlinearity $f(t,x)$ is jointly continuous (in particular then the potential $F(t,x)=\int_0^x f(t,r)\,\mathrm{d}r$ is C^1) and they employ conditions on the interaction of f with the Fučik spectrum of the differential operator.

Our hypotheses on nonsmooth potential $j(t,x)$ are the following:

$\mathrm{H}(j)_1$: $j: T\times\mathbb{R}^N\to\mathbb{R}$ is a function such that $j(t,0)=0$ a.e. on T and

(i) for all $x\in\mathbb{R}^N$, $t\to j(t,x)$ is measurable;

(ii) for almost all $t\in T$, $x\to j(t,x)$ is locally Lipschitz;

(iii) for almost all $t\in T$, all $x\in\mathbb{R}^N$ and all $u\in\partial j(t,x)$ we have

$$\|u\|\leqslant a(t)(1+\|x\|^{\theta}) \text{ with } a\in L^1(T) \text{ and } 0\leqslant\theta<p-1;$$

(iv) $\dfrac{1}{\|x\|^{\theta q}}\displaystyle\int_0^b j(t,x)\,\mathrm{d}t\to+\infty$ as $\|x\|\to\infty$, $x\in\mathbb{R}^N$, $\dfrac{1}{p}+\dfrac{1}{q}=1$;

(v) for almost all $t\in T$ and all $\|x\|\leqslant 1$, we have $j(t,x)\geqslant -\frac{1}{b^p p}\|x\|^p$;

(vi) there exists $\widehat{\delta}>0$ such that for almost all $t\in T$ and all $\|x\|\leqslant\widehat{\delta}$, we have $j(t,x)\leqslant 0$ and there exists $c_0\in\mathbb{R}^N$ such that $j(t,c_0)<0$ for almost all $t\in T$.

Let $W^{1,p}_{per}(T,\mathbb{R}^N)=\{x\in W^{1,p}(T,\mathbb{R}^N): x(0)=x(b)\}$ and let ϕ: $W^{1,p}_{per}(T,\mathbb{R}^N)\to\mathbb{R}$ be defined by

$$\phi(x)=\frac{1}{p}\|x'\|_p^p+\int_0^b j(t,x(t))\ \mathrm{d}t.$$

We know that ϕ is locally Lipschitz (in the sense of being Lipschitz on bounded sets, see [17], p. 313).

PROPOSITION 9. *If hypotheses* $\mathrm{H}(j)_1$ *hold, then* ϕ *satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ be a sequence such that

$$|\phi(x_n)| \leqslant M_1 \text{ for some } M_1 > 0 \text{ and all } n \geqslant 1 \text{ and } m(x_n) \to 0.$$

Let $x_n^* \in \partial\phi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geqslant 1$. Also let $A : W^{1,p}_{per}(T, \mathbb{R}^N) \to W^{1,p}_{per}(T, \mathbb{R}^N)^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b \|x'(t)\|^{p-2}(x'(t), y'(t))_{\mathbb{R}^N}\,\mathrm{d}t \quad \text{for all } x, y \in W^{1,p}_{per}(T, \mathbb{R}^N).$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}_{per}(T, \mathbb{R}^N), W^{1,p}_{per}(T, \mathbb{R}^N)^*)$. It is easy to see that A is demicontinuous, monotone, thus maximal monotone. We have

$$x_n^* = A(x_n) + u_n, \quad n \geqslant 1$$

with $u_n \in L^{\theta'}(T, \mathbb{R}^N)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see [6], p. 47).

We claim that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ is bounded. To this end, we consider the direct sum decomposition $W^{1,p}_{per}(T, \mathbb{R}^N) = \mathbb{R}^N \oplus V$, where $V = \{v \in W^{1,p}_{per}(T, \mathbb{R}^N) : \int_0^b v(t)\,\mathrm{d}t = 0\}$. So given $x \in W^{1,p}_{per}(T, \mathbb{R}^N)$ we write in a unique way $x = \overline{x} + \hat{x}$ with $\overline{x} \in \mathbb{R}^N$, $\hat{x} \in V$. From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ we have

$$|\langle A(x_n), \hat{x}_n\rangle + (u_n, \hat{x}_n)_{\theta\theta'}| \leqslant \varepsilon_n\|\hat{x}_n\| \quad \text{with } \varepsilon_n \downarrow 0. \tag{5}$$

Here by $(\cdot,\cdot)_{\theta,\theta'}$ we denote the duality brackets for the pair $(L^{\theta}(T, \mathbb{R}^N), L^{\theta'}(T, \mathbb{R}^N))$. From hypothesis $\mathrm{H}(j)_1$(iii), we have that

$$\begin{aligned}
\left|(u_n(t), \hat{x}_n(t))_{\mathbb{R}^N}\right| &\leqslant a(t)(1 + \|\overline{x}_n + \hat{x}_n(t)\|^{\theta})\|\hat{x}_n(t)\| \\
&\leqslant a(t)\|\hat{x}_n(t)\| + 2^{\theta-1}\|\overline{x}_n\|^{\theta}\|\hat{x}_n(t)\| \\
&\quad + 2^{\theta-1}\|\hat{x}_n(t)\|^{\theta+1} \text{ a.e. on } T,
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left|\int_0^b (u_n(t), \hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t\right| \\
&\leqslant \|a\|_1\|\hat{x}_n\|_\infty + 2^{\theta-1}b\|\hat{x}_n\|_\infty^{\theta+1} + 2^{\theta-1}b\left(\frac{\varepsilon}{p}\|\hat{x}_n\|_\infty^p + \frac{1}{\varepsilon q}\|\overline{x}_n\|^{\theta q}\right) \\
&\leqslant \beta_1\|\hat{x}_n'\|_p + \beta_2\|\hat{x}_n'\|_p^{\theta+1} + \beta_3\frac{\varepsilon}{p}\|\hat{x}_n'\|_p^p + \beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}
\end{aligned} \tag{6}$$

for $\varepsilon > 0$ and some $\beta_1, \beta_2, \beta_3, \beta_4(\varepsilon) > 0$. In obtaining (6) we have used the Poincaré–Wirtinger inequality (see [26], p. 8). Note that $\langle A(x_n), \hat{x}_n \rangle = \|\hat{x}_n'\|_p^p$. So returning to (5) and using these facts, we obtain

$$\|\hat{x}_n'\|_p^p - \beta_1 \|\hat{x}_n'\|_p - \beta_2 \|\hat{x}_n'\|_p^{\theta+1} - \beta_3 \frac{\varepsilon}{p} \|\hat{x}_n'\|_p^p - \beta_4(\varepsilon) \|\overline{x}_n\|^{\theta q} \leqslant M_2 \|\hat{x}_n\|$$

for some $M_2 > 0$ and all $n \geqslant 1$, hence

$$\left(1 - \beta_3 \frac{\varepsilon}{p}\right) \|\hat{x}_n'\|_p^p - (\beta_1 + M_3) \|\hat{x}_n'\|_p - \beta_2 \|\hat{x}_n'\|_p^{\theta+1} \leqslant \beta_4(\varepsilon) \|\overline{x}_n\|^{\theta q} \tag{7}$$

for some $M_3 > 0$ and all $n \geqslant 1$. Again we have used the Poincaré–Wirtinger inequality. Let $\varepsilon > 0$ be small enough so that $\beta_3 \frac{\varepsilon}{p} < 1$. We shall show that

$$\|\hat{x}_n'\|_p^{p-1} \leqslant \beta_5 \|\overline{x}_n\|^{\theta} + \beta_6 \quad \text{for some } \beta_5, \beta_6 > 0 \quad \text{and all } n \geqslant 1. \tag{8}$$

Indeed if $\{\hat{x}_n\}_{n \geqslant 1} \subseteq W_{per}^{1,p}(T, \mathbb{R}^N)$ is bounded, then (8) is clear. Otherwise suppose that $\|\hat{x}_n\| \to \infty$ hence $\|\hat{x}_n'\|_p \to +\infty$. Then from (7) we have

$$\beta_7 \|\hat{x}_n'\|_p^p - \beta_8 \|\hat{x}_n'\|_p^{\theta+1} \leqslant \beta_4(\varepsilon) \|\overline{x}_n\|^{\theta q} \quad \text{for some } \beta_7, \beta_8 > 0 \quad \text{and all } n \geqslant 1$$

$$\Rightarrow \beta_9 \|\hat{x}_n'\|_p^p \leqslant \beta_4(\varepsilon) \|\overline{x}_n\|^{\theta q} + \beta_{10} \quad \text{for some } \beta_9, \beta_{10} > 0 \quad \text{and all } n \geqslant 1.$$

From this and since $(p/q) = p - 1$, we obtain (8).

Let $S_n(t) = \{(u, \lambda) \in \mathbb{R}^N \times (0,1) : u \in \partial j(t, \overline{x}_n + \lambda \hat{x}_n(t)), j(t, \overline{x}_n + \hat{x}_n(t)) - j(t, \overline{x}_n) = (u, \hat{x}_n(t))_{\mathbb{R}^N}\}$. From Lebourg's mean value theorem (see [6], p. 41, and [22]), we know that for almost all $t \in T$, $S_n(t) \neq \emptyset$. By redefining S_n on the exceptional Lebesgue-null set, we may assume that $S_n(t) \neq \emptyset$ for all $t \in T$. We claim that for every $h \in \mathbb{R}^N$, the function $(t, \lambda) \to j^0(t, \overline{x}_n + \lambda \hat{x}_n(t); h)$ is measurable on $T \times (0,1)$. To this end, from the definition of the generalized directional derivative, we have

$$\begin{aligned} & j^0(t, \overline{x}_n + \lambda \hat{x}_n(t); h) \\ & \quad = \inf_{m \geqslant 1} \sup \left[\frac{j(t, \overline{x}_n + \lambda \hat{x}_n(t) + r + sh) - j(t, \overline{x}_n + \lambda \hat{x}_n(t) + r)}{s} : \right. \\ & \qquad \qquad \left. r \in \mathcal{Q}^N \cap B_{(1/m)}(0), s \in \mathcal{Q} \cap \left(0, \frac{1}{m}\right) \right] \\ & \Rightarrow (t, \lambda) \to j^0(t, \overline{x}_n + \lambda \hat{x}_n(t); h) \text{ is measurable on } T \times (0,1). \end{aligned}$$

Set $G_n(t,\lambda)=\partial j(t,\overline{x}_n+\lambda\hat{x}_n(t))$ and $\{h_m\}_{m\geqslant 1}\subseteq \mathbb{R}^N$ be a dense sequence. Since $j^0(t,\overline{x}_n+\lambda_n\hat{x}_n(t);\cdot)$ is continuous, we have

$$
\begin{aligned}
GrG_n &= \cap_{m\geqslant 1}\{(t,\lambda,u)\in T\times(0,1)\times\mathbb{R}^N:(u,h_m)_{\mathbb{R}^N}\\
&\qquad \leqslant j^0(t,\overline{x}_n+\lambda\hat{x}_n(t);h_m)\}\\
&\qquad \Rightarrow GrG_n\in\mathcal{L}(T)\times\mathcal{B}(I)\times\mathcal{B}(\mathbb{R}^N)
\end{aligned}
$$

with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $\mathcal{B}(I)$ (resp $\mathcal{B}(\mathbb{R}^N)$) the Borel σ-field of $I=(0,1)$ (resp. of $\mathbb{R}^N$). So we can apply the Yankov-von Neumann–Aumann selection theorem (see Hu-Papageorgiou [16], p. 158) to obtain Lebesgue measurable maps $u_n:T\to\mathbb{R}^N$ and $\lambda_n:T\to I$ such that $(u_n(t),\lambda_n(t))\in S_n(t)$ for all $n\geqslant 1$. So we have

$$
\begin{aligned}
\phi(x_n) &= \frac{1}{p}\|x_n'\|_p^p+\int_0^b j(t,x_n(t))\,\mathrm{d}t\\
&= \frac{1}{p}\|x_n'\|_p^p+\int_0^b j(t,\overline{x}_n+\hat{x}_n(t))\,\mathrm{d}t-\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t+\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t\\
&= \frac{1}{p}\|\hat{x}_n'\|_p^p+\int_0^b (u_n(t),\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t+\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t
\end{aligned}
$$

with $u_n(t)\in\partial j(t,\overline{x}_n+\lambda_n(t)\hat{x}_n(t))$ a.e. on T. From the choice of the sequence $\{x_n\}_{n\geqslant 1}\subseteq W^{1,p}_{per}(T,\mathbb{R}^N)$, we have $\phi(x_n)\leqslant M_1$ for all $n\geqslant 1$. So

$$
\frac{1}{p}\,\|\hat{x}_n'\|_p^p+\int_0^b (u_n(t),\hat{x}_n(t))_{\mathbb{R}^N}\,\mathrm{d}t+\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t\leqslant M_1\ .
$$

Using (6) we have

$$
\begin{aligned}
&\frac{1}{p}\|\hat{x}_n'\|_p^p-\beta_1\|\hat{x}_n'\|_p-\beta_2\|\hat{x}_n'\|_p^{\theta+1}\\
&\quad -\beta_3\frac{\varepsilon}{p}\|\hat{x}_n'\|_p^p-\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}+\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t\leqslant M_1\\
&\Rightarrow \left(\frac{1}{p}-\beta_3\frac{\varepsilon}{p}\right)\|\hat{x}_n'\|_p^p-\beta_1\|\hat{x}_n'\|_p-\beta_2\|\hat{x}_n'\|_p^{\theta+1}\\
&\quad -\beta_4(\varepsilon)\|\overline{x}_n\|^{\theta q}+\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t\leqslant M_1\ .
\end{aligned}
$$

Choose $\varepsilon > 0$ so that $\varepsilon < (1/\beta_3)$. From the last inequality we have

$$\begin{aligned}
&\beta_{11}\|\hat{x}_n'\|_p^p - \beta_1\|\hat{x}_n'\|_p - \beta_2\|\hat{x}_n'\|_p^{\theta+1} \\
&\quad + \|\overline{x}_n\|^{\theta q}\left(\frac{1}{\|\overline{x}_n\|^{\theta q}}\int_0^b j(t,\overline{x}_n)\,\mathrm{d}t - \beta_4(\varepsilon)\right) \leqslant M_1 \qquad (9)
\end{aligned}$$

for some $\beta_{11} > 0$ and all $n \geqslant 1$. If $\{\hat{x}_n'\}_{n\geqslant 1} \subseteq L^p(T, \mathbb{R}^N)$ is unbounded, then we may assume that $\|\hat{x}_n'\|_p \to \infty$ and so from (8) we have that $\|\overline{x}_n\| \to \infty$. So if we pass to the limit in (9) and using hypothesis $\mathrm{H}(j)_1$(iv) and the fact that $\theta + 1 < p$, we reach a contradiction. Hence $\{\hat{x}_n'\}_{n\geqslant 1} \subseteq L^p(T, \mathbb{R}^N)$ is bounded. Suppose that $\|\overline{x}_n\| \to \infty$. Then again from (9) by passing to the limit and using hypothesis $\mathrm{H}(j)_1$(iv), we have a contradiction. Therefore we infer that $\{x_n\} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ is bounded. Thus we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$ and $x_n \to x$ in $C(T, \mathbb{R}^N)$. We have

$$\langle A(x_n), x_n - x\rangle + (u_n, x_n - x)_{\theta\theta'} \leqslant \varepsilon_n\|x_n - x\| \leqslant \varepsilon_n M_3$$

for some $M_3 > 0$ and all $n \geqslant 1$. Evidently $\{u_n\}_{n\geqslant 1} \subseteq L^{\theta'}(T, \mathbb{R}^N)$ is bounded. So $(u_n, x_n - x)_{\theta\theta'} \to 0$, hence

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leqslant 0.$$

But A being maximal monotone, is generalized pseudomonotone and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Rightarrow \|x_n'\|_p \to \|x'\|_p$. Since $x_n' \xrightarrow{w} x'$ in $L^p(T, \mathbb{R}^N)$ and the latter is uniformly convex, from the Kadec-Klee property we conclude that $x_n' \to x'$ in $L^p(T, \mathbb{R}^N)$, hence $x_n \to x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$. □

PROPOSITION 10. *If hypotheses* $\mathrm{H}(j)_1$ *hold, then* ϕ *is bounded below.*

Proof. For every $x \in W^{1,p}_{per}(T, \mathbb{R}^N)$, we have (see the proof of Proposition 9)

$$\begin{aligned}
\phi(x) &= \frac{1}{p}\|x'\|_p^p + \int_0^b (j(t, \overline{x} + \hat{x}(t)) - j(t, \overline{x}))\,\mathrm{d}t + \int_0^b j(t, \overline{x})\,\mathrm{d}t \\
&= \frac{1}{p}\|x'\|_p^p + \int_0^b (u(t), \hat{x}(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_0^b j(t, \overline{x})\,\mathrm{d}t \\
&\qquad (\text{with } u \in L^{\theta'}(T, \mathbb{R}^N),\ u(t) \in \partial j(t, \overline{x} + \lambda(t)\hat{x}(t)) \text{ a.e.})
\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{1}{p}\|\hat{x}'\|_p^p - \beta_1\|\hat{x}'\|_p - \beta_2\|\hat{x}'\|_p^{\theta+1} - \beta_3\frac{\varepsilon}{p}\|\hat{x}'\|_p^p - \beta_4(\varepsilon)\|\overline{x}\|^{\theta q} \\
&\quad + \int_0^b j(t,\overline{x})\,\mathrm{d}t \ \ \text{(see (6))} \\
&= \left(\frac{1}{p} - \beta_3\frac{\varepsilon}{p}\right)\|\hat{x}'\|_p^p - \beta_1\|\hat{x}'\|_p - \beta_2\|\hat{x}'\|_p^{\theta+1} \\
&\quad + \|\overline{x}\|^{\theta q}\left(\frac{1}{\|\overline{x}\|^{\theta q}}\int_0^b j(t,\overline{x})\,\mathrm{d}t - \beta_4(\varepsilon)\right).
\end{aligned}$$

Choosing $\varepsilon < (1/\beta_3)$ and since $\theta + 1 < p$, from the last inequality it follows that ϕ is coercive, hence it is bounded from below. □

Using Propositions 9 and 10, we can have the following multiplicity result for problem (4).

THEOREM 11. *If hypotheses* $\mathrm{H}(j)_1$ *hold, then problem* (4) *has at least two distinct nontrivial solutions* $x_1, x_2 \in C^1(T, \mathbb{R}^N)$.

Proof. From hypothesis $\mathrm{H}(j)_1$(vi) we have that $\inf \phi < 0$. Also from hypothesis $\mathrm{H}(j)_1$(v), for almost all $t \in T$ and all $\|x\| \leqslant 1$ we have $-(1/b^p p)\|x\|^p \leqslant j(t,x)$. Again we consider the direct sum decomposition $W^{1,p}_{per}(T,\mathbb{R}^N) = \mathbb{R}^N \oplus V$. Let $v \in V$ with $\|v'\| \leqslant \frac{1}{b^{(1/q)}}$. Recall that $\|v\|_\infty \leqslant b^{\frac{1}{q}}\|v'\|_p \leqslant 1$ (Poincaré–Wirtinger inequality, see [26],p. 8). So for $v \in V$ with $\|v\| = (\|v\|_p^p + \|v'\|_p^p)^{\frac{1}{p}} \leqslant \frac{1}{b^{1/q}} = \delta$, we have $\|v\|_\infty \leqslant 1$ and so

$$\begin{aligned}
\phi(v) &= \frac{1}{p}\|v'\|_p^p + \int_0^b j(t,v(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|v'\|_p^p - \frac{1}{b^{p-1}p}\|x\|^p \\
&\geqslant \frac{1}{p}\|v'\|_p^p - \frac{1}{b^p p} b\|v\|_\infty^p \\
&\geqslant \frac{1}{p}\|v'\|_p^p - \frac{1}{p}\|v'\|_p^p = 0.
\end{aligned}$$

If $0 < \varepsilon < \beta_1$, we have $\phi(v) \geqslant 0$ for all $v \in V$ with $\|v\| \leqslant \delta_1$.

In addition by virtue of hypothesis $\mathrm{H}(j)_1$(vi), we can find $\delta_2 > 0$ such that if $x \in \mathbb{R}^N \subseteq W^{1,p}_{per}(T,\mathbb{R}^N)$ and $\|x\| \leqslant \delta_2$, then $\phi(x) \leqslant 0$. Thus if $\delta_3 = \min\{\delta_1, \delta_2\}$, we have

$$\phi(v)\geqslant 0 \quad \text{for all } v\in V,\ \|v\|\leqslant\delta_3 \text{ and } \phi(y)\leqslant 0 \quad \text{for all } y\in\mathbb{R}^N,\ \|y\|\leqslant\delta_3.$$

Therefore, we can apply Theorem 8 and obtain $x_1\neq x_2, x_1, x_2\neq 0$ such that $0\in\partial\phi(x_k), k=1,2$. Let $y=x_k, k=1,2$. We have

$$A(y)=u \text{ with } u\in L^{\theta'}(T,\mathbb{R}^N),\ u(t)\in\partial j(t,y(t)) \text{ a.e. on } T,$$

$$\Rightarrow \langle A(y),\psi\rangle=(u,\psi)_{\theta\theta'} \quad \text{for all} \quad \psi\in C_0^\infty((0,b),\mathbb{R}^N)$$

$$\Rightarrow \int_0^b \|y'(t)\|^{p-2}(y'(t),\psi'(t))_{\mathbb{R}^N}\,\mathrm{d}t=\int_0^b (u(t),\psi(t))_{\mathbb{R}^N}\,\mathrm{d}t$$

$$\Rightarrow -(\|y'(t)\|^{p-2}y'(t))'=u(t) \text{ a.e. on } T,\ \ y(0)=y(b) \tag{10}$$

Hence $\|y'(\cdot)\|^{p-2}y'(\cdot)\in W^{1,\theta}(T,\mathbb{R}^N)\subseteq C(T,\mathbb{R}^N)$ and since the map ξ : $\mathbb{R}^N\to\mathbb{R}^N$ defined by $\xi(x)=\|x\|^{p-2}x$ is a homeomorphism, it follows that $y'\in C(T,\mathbb{R}^N)$, hence $y\in C^1(T,\mathbb{R}^N)$. Also if $v\in W^{1,p}_{per}(T,\mathbb{R}^N)$, then by integration by parts and using (10), we obtain

$$\|y'(0)\|^{p-2}(y'(0),v'(0))_{\mathbb{R}^N}=\|y'(b)\|^{p-2}(y'(b),v'(b))_{\mathbb{R}^N}$$
$$\text{for all } v\in W^{1,p}_{per}(T,\mathbb{R}^N)$$

$$\Rightarrow \|y'(0)\|^{p-2}y'(0)=\|y'(b)\|^{p-2}y'(b)$$

$$\Rightarrow y'(0)=y'(b).$$ □

Remark. Let $p>3$ and consider the following nonsmooth locally Lipschitz in $x\in\mathbb{R}^{\mathbb{N}}$ function:

$$j(t,x)=\begin{cases}-\frac{1}{b^p p}\|x\|^p & \text{if } \|x\|\leqslant 1,\\ \eta(t)\|x\|^{\frac{3}{2}}-\eta(t)-\frac{1}{b^p p} & \text{if } \|x\|>1,\end{cases},\ \ \eta\in L^1(T)_+.$$

Then $j(t,x)$ satisfies hypotheses $H(j)_1$.

Next we consider a Dirichlet problem for nonlinear hemivariational inequalities at resonance. So let $Z\subseteq\mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. We examine the following problem:

$$\left\{\begin{array}{l}-div\left(\|Dx(z)\|^{p-2}Dx(z)\right)-\lambda_1|x(z)|^{p-2}x(z)\in\partial j(z,x(z)) \text{ a.e. on } Z\\ x|_\Gamma=0,\ 2\leqslant p<\infty.\end{array}\right\}. \tag{11}$$

Again $\partial j(z,x)$ is generalized subdifferential of the generally nonsmooth locally Lipschitz potential $j(z,x)$. Our hypotheses on $j(z,x)$ are the following:

$H(j)_2$: $j\colon Z\times\mathbb{R}\to\mathbb{R}$ is a function such that $j(z,0)=0$ a.e. on Z and

(i) for all $x\in\mathbb{R}$, $z\to j(z,x)$ is measurable;
(ii) for almost all $z\in Z$, $x\to j(z,x)$ is locally Lipschitz;
(iii) for almost all $z\in Z$, all $x\in\mathbb{R}$ and all $u\in\partial j(z,x)$ we have

$$|u|\leqslant a(z)+c|x|^{p-1} \text{ with } a\in L^\infty(Z),\ c>0;$$

(iv) there exist $\beta>0$ and $0<\mu<p$ such that

$$\liminf_{|x|\to\infty}\frac{ux-pj(z,x)}{|x|^\mu}>\beta \text{ or } \limsup_{|x|\to\infty}\frac{ux-pj(z,x)}{|x|^\mu}<-\beta$$

uniformly for almost all $z\in Z$ and all $u\in\partial j(z,x)$ and $j(z,x)\leqslant\gamma(z)$ a.e. on Z for $|x|\geqslant M$, $\gamma\in L^1(Z)$;
(v) $\limsup_{x\to 0}\frac{pj(z,x)}{|x|^p}\leqslant 0$ uniformly for almost all $z\in Z$;
(vi) there exists $\delta>0$ such taht for all $|x|\leqslant\delta$ we have $\int_Z j(z,x)\,dz\geqslant 0$ and there exists $\xi\in\mathbb{R}$ such that $\int_Z j(z,\xi u_1(z))\,dz>0$.

We consider the functional $\phi\colon W_0^{1,p}(Z)\to\mathbb{R}$ defined by

$$\phi(x)=\frac{1}{p}\|Dx\|_p^p-\frac{\lambda_1}{p}\|x\|_p^p-\int_Z j(z,x(z))\,dz.$$

Again ϕ is locally Lipschitz (in the sense of being Lipschitz on bounded sets).

PROPOSITION 12. *If hypotheses $H(j)_2$ hold, then ϕ satisfies the nonsmooth C-condition.*

Proof. We assume that in hypothesis $H(j)_2$(iv) the first alternative holds. The proof is similar if the second alternative is in effect.

So let $\{x_n\}_{n\geqslant 1}\subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\phi(x_n)|\leqslant M_1 \text{ for some } M_1>0 \quad\text{and all } n\geqslant 1$$
$$\text{and } (1+\|x_n\|)m(x_n)\to 0 \text{ as } n\to\infty.$$

Let $x_n^*\in\partial\phi(x_n)$ be such that $m(x_n)=\|x_n^*\|$, $n\geqslant 1$. We have $x_n^*=A(x_n)-\lambda_1|x_n|^{p-2}-u_n$, with $A: W_0^{1,p}(Z)\to W^{-1,q}(Z)$ being defined by $\langle A(x),y)\rangle=\int_Z\|Dx(z)\|^{p-2}(Dx(z),Dy(z))_{\mathbb{R}^N}\,dz$ for all $x,y\in W_0^{1,p}(Z)$ and

$u_n \in L^q(Z)$, $u_n(z) \in \partial j(z, x_n(z))$ a.e. on Z (here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $\left(W_0^{1,p}(Z), W^{-1,q}(Z)\right)$). Because $(1 + \|x_n\|)m(x_n) \to 0$, we can say that $| < x_n^*, x_n > | \leqslant (1/n)$. So

$$-\|Dx_n\|_p^p + \lambda_1 \|x_n\|_p^p + (u_n, x_n)_{pq} \leqslant \frac{1}{n}. \tag{12}$$

Also, since $|p\phi(x_n)| \leqslant pM_1$ for all $n \geqslant 1$, we have

$$\|Dx_n\|_p^p - \lambda_1 \|x_n\|_p^p - p \int_Z j(z, x_n(z))\, \mathrm{d}z \leqslant pM_1. \tag{13}$$

Adding (12) and (13), we obtain

$$\int_Z (u_n(z), x_n(z) - pj(z, x_n(z)))\, \mathrm{d}z \leqslant \frac{1}{n} + pM_1. \tag{14}$$

By virtue of hypothesis $\mathrm{H}(j)_2$(iv), we can find $M_2 > 0$ such that for almost all $z \in Z$, all $|x| \geqslant M_2$ and all $u \in \partial j(z, x)$, we have

$$\frac{\beta}{2}|x|^\mu < ux - pj(z, x). \tag{15}$$

Using the Lebourg mean value theorem and hypothesis $\mathrm{H}(j)_2$(iii), we see that for amost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|j(z, x)| \leqslant a_1(z)(1 + |x|^p) \qquad \text{with } a_1 \in L^\infty(Z). \tag{16}$$

From (16) and hypothesis $\mathrm{H}(j)_2$(iii) it follows that for almost all $z \in Z$, all $|x| < M_2$ and all $u \in \partial j(z, x)$, we have

$$|ux - pj(z, x)| \leqslant a_2(z) \quad \text{with } a_2 \in L^\infty(Z). \tag{17}$$

So from (15) and (17), we see that for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have

$$\frac{\beta}{2}|x|^\mu - a_3(z) \leqslant ux - pj(z, x) \quad \text{with } a_3 \in L^\infty(Z).$$

Returning to (14), we have

$$\frac{\beta}{2}\|x_n\|_\mu^\mu \leqslant \int_Z (u_n(z)x_n(z) - pj(z, x_n(z)))\, \mathrm{d}z + \|a_3\|_1 \leqslant \frac{1}{n} + pM_1 + \|a_3\|_1$$

$$\Rightarrow \|x_n\|_\mu \leqslant c_1 \text{ for all } n \geqslant 1, \text{ i.e. } \{x_n\}_{n \geqslant 1} \subseteq L^\mu(Z) \text{ is bounded.} \tag{18}$$

As usual, let p^* be the Sobolev critical exponent, i.e.

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leqslant p \end{cases}.$$

Choose $r \geqslant 1$ such that

$$p < r < \min\left\{p^*, p\frac{\max\{N, p\} + \mu}{\max\{N, p\}}\right\}.$$

From (16) we see that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have that

$$j(z, x) \leqslant c_2 + c_3|x|^r \text{ with } c_2, c_3 > 0. \tag{19}$$

Set $\theta = \begin{cases} \frac{p^*(r-\mu)}{r(p^*-\mu)} & \text{if } N > p, \\ 1 - \frac{\mu}{r} & \text{if } N \leqslant p. \end{cases}$

Note that $0 < \theta < 1$ and $(1/r) = (1-\theta)/\mu + (\theta/p^*)$. So from the interpolation inequality (see [2], p. 57), from (18) and from the Sobolev embedding theorem, we obtain

$$\|x_n\|_r \leqslant \|x_n\|_\mu^{1-\theta}\|x_n\|_{p^*}^\theta \leqslant c_4\|x_n\|^\theta \quad \text{with } c_4 > 0. \tag{20}$$

Recall that $\phi_\lambda(x_n) \leqslant M_1$ for all $n \geqslant 1$. Using this fact, inequality (19), the continuity of the embedding $L^r(Z)$ into $L^p(Z)$ (since $p < r$), Young's inequality and (20), we obtain

$$\begin{aligned}
\frac{1}{p}\|Dx_n\|_p^p &\leqslant \frac{\lambda_1}{p}\|x_n\|_p^p + c_2|Z| + c_3\|x_n\|_r^r + M_1 \\
&\leqslant c_5\|x_n\|_r^p + c_2|Z| + c_3\|x_n\|_r^r + M_1 \\
&\leqslant c_6 + c_7\|x_n\|_r^r + c_2|Z| + c_3\|x_n\|_r^r + M_1 \\
&\leqslant c_8 + c_9\|x_n\|_r^r \leqslant c_8 + c_{10}\|x_n\|^{\theta r}
\end{aligned}$$

for some $c_5, c_6, c_7, c_8, c_9, c_{10} > 0$. From this and Poincaré's inequality, we obtain

$$\|Dx_n\|_p^p \leqslant c_{11}\|Dx_n\|_p^{\theta r} + c_{12} \quad \text{with } c_{11}, c_{12} > 0. \tag{21}$$

If $N > p$, then $Nr < Np + \mu p$ and so

$$\theta r = \frac{p^*(r-\mu)}{p^*-\mu} = \frac{Np}{N-p}\frac{(r-\mu)(N-p)}{Np - N\mu + \mu p} < \frac{Np}{N-p}\frac{(r-\mu)(N-p)}{Nr - N\mu} = p.$$

If $N \leqslant p$, then $r < \min\left\{p^*, p\frac{\max\{N,p\}+\mu}{\max\{N,p\}}\right\} = p((p+\mu)/(p)) = p+\mu$ and so $r-\mu < p$. Hence

$$\theta r = \left(1-\frac{\mu}{r}\right) r = r-\mu < p.$$

So in both cases we have seen that $\theta r < p$. Therefore from (21) and Poincaré's inequality it follows that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. Arguing as in the proof of Proposition 9, via the generalized pseudomonotonicity of A (being maximal monotone) and the Kadec–Klee property, we obtain that $x_n \to x$ in $W_0^{1,p}(Z)$. $\square$

Consider the direct sum decomposition

$$W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus V$$

with $V = \{v \in W_0^{1,p}(Z) : \int_Z v u_1^{p-1}\, dz = 0\}$. Recall that u_1 is the normalized principal eigenfunction of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$ and $u_1(z) > 0$ for all $z \in Z$

PROPOSITION 13. *If hypotheses* $\mathrm{H}(j)_2$ *hold, then there exists* $\delta_1 > 0$ *such that if* $v \in V$, $\|v\| \leqslant \delta_1$, *then* $\phi(v) \geqslant 0$.

Proof. By virtue of hypothesis $H(j)_2(v)$, given $\varepsilon > 0$ we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant \delta$ we have $|j(z,x)| \leqslant c_1|x|^r$. So finally for almost all $z \in Z$ and all $x \in \mathbb{R}$, we can write that $j(z,x) \leqslant \frac{\varepsilon}{p}|x|^p + c_1|x|^r$. For every $v \in V$ we have

$$\begin{aligned}\phi(v) &= \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p - \int_Z j(z, v(z))\, dz \\ &\geqslant \frac{1}{p}\left(1-\frac{\lambda_1}{\lambda_V}-\frac{\varepsilon}{\lambda_V}\right)\|Dv\|_p^p - c_2\|Dv\|_p^r,\end{aligned}$$

where $\lambda_V = \inf\left\{\frac{\|Dv\|_p^p}{\|v\|_p^p} : v \in V,\ v \neq 0\right\}$. Since λ_1 is isolated, $\lambda_1 < \lambda_V$. So we can choose $\varepsilon > 0$ small so that $\lambda_1 + \varepsilon < \lambda_V$. Hence we have

$$\phi(v) \geqslant c_3\|Dv\|_p^p - c_2\|Dv\|_p^r$$

$$\geqslant c_4\|v\|^p - c_5\|v\|^r \text{ for some } c_4, c_5 > 0 \text{ (by Poincaré's inequality).}$$

Since $r > p$, we can find $\delta_1 > 0$ such that if $\|v\| \leqslant \delta_1$, then $\phi(v) \geqslant 0$. $\square$

THEOREM 14. *If hypotheses* $H(j)_2$ *hold, then problem* (11) *has at least two distinct nontrivial solutions.*

Proof. Let $\delta_2=(\delta/\|u_1\|_\infty)$ and $|\theta|\leqslant\delta_2$. Then $|\theta u_1(z)|\leqslant\delta$ for all $z\in Z$ and so by hypothesis $\mathrm{H}(j)_2$(vi) and since $\|D(\theta u_1)\|_p^p=\lambda_1\|\theta u_1\|_p^p$, we have that $\phi(\theta u_1)\leqslant 0$. Also $\phi(\xi u_1)\leqslant 0$, so $\inf\phi<0$. Moreover from the last part of hypothesis $\mathrm{H}(j)_2$(iv), we see that ϕ is bounded below.

Finally, let $\delta_3=\min\{\delta_1,\delta_2\}$. Then we have

$$\phi(x)=\begin{cases}\geqslant 0 & \text{if } \|x\|\leqslant\delta_3,\ x\in V,\\ \leqslant 0 & \text{if } \|x\|\leqslant\delta_3,\ x\in\mathbb{R}u_1,\end{cases}\quad \text{local linking.}$$

Therefore, we can apply Theorem 8 and obtain two distinct, nontrivial critical points of ϕ. Easily we check that these are distinct nontrivial solutions of (11). □

Remark. If $a\in L^\infty(Z)_+,\ 2<p<5$ and

$$j(z,x)=\begin{cases}a(z)x^5 & \text{if } |x|<1,\\ a(z)(2|x|-x^2)+\sin x-\sin 1 & \text{if } |x|\geqslant 1,\end{cases}$$

then hypotheses $\mathrm{H}(j)_2$ are satisfied (the first alternative in $\mathrm{H}(j)_2$(iv)). In this case we can take $\mu=2$.

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References

1. Bartolo, P., Benci, V. and Fortunato, D. (1983), Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, *Nonlinear Analyses* 7, 981–1012.
2. Brezis, H. (1983), *Analyse Fonctionnelle*, Masson, Paris.
3. Brezis, H. and Nirenberg, L. (1991), Remarks on finding critical points, *Communication Pure Applied Mathematics* 44, 939–963.
4. Cerami, G. (1978), Un criterio di esistenza per i punti critici su varieta illimitate, Rends Academic Science Letters Ist Lobardo, 112, 332–336.
5. Chang, K.-C. (1981), Variational methods for non-differentiable functionals and their applications to partial differential equations, *Journal of Math. Anal. Applic.* 80, 102–129.
6. Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis*, Wiley, New York.
7. Costa, D. and Goncalves, J. (1990), Critical point theory for nondifferentiable functionals and applications, *Journal of Math. Anal. Applic.* 153, 470–485.
8. Degiovanni, M., Marzocchi, M. and Radulescu, V. (2000), Multiple solutions of hemivariational inequalities with area-type term, *Calculus of Variations* 10, 355–387.

9. Del Pino, M. Manasevich, R. and Murua, A. (1992), Existence and multiplicity of solutions with prescribed period for a second order quasilinear ode, *Nonlinear Analysis* 18, 79–92.
10. Drabek, P. and Invernizzi, S. (1986), On the periodic bvp for the forced Duffing equation with jumping nonlinearity, *Nonlinear Analysis* 10, 643–650.
11. Fabry, C., Mawhin, J. and Nkashama, M.N. (1986), A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations, Bulletin London Mathematical Society. 18, 173–180.
12. Gasinski, L. and Papageorgiou, N.S. (2000), Multiple solutions for nonlinear hemivariational inequalities near resonance, *Funkcialaj Ekvacioj* 43, 271–284.
13. Gasinski, L. and Papageorgiou, N.S. (2001), Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance, Proceeding Royal Society Edinburgh (Math) 131A, 1–21.
14. Goeleven, D., Motreanu, D. and Panagiotopoulos, P. (1998), Eigenvalue problems for variational-hemivariational inequalities at resonance, *Nonlinear Analysis* 33, 161–180.
15. Hu, S., Kourogenis, N.C. and Papageorgiou, N.S. (1999), Nonlinear elliptic eigenvalue problems with discontinuities, *Journal of Mathematical Analysis and Applications* 233, 406–424.
16. Hu, S. and Papageorgiou, N.S. (1997), *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, The Netherlands.
17. Hu, S. and Papageorgiou, N.S. (2000), *Handbook of Multivalued Analysis, Volume II: Applications*, Kluwer, Dordrecht, The Netherlands.
18. Kourogenis, N.C. and Papageorgiou, N.S. (2000), Nonsmooth critical point theory and nonlinear elliptic equations at resonance, *Journal of Australian Mathematical Society* 69, 245–271.
19. Kourogenis, N.C. and Papageorgiou, N.S. (2000), A weak nonsmooth Palais-smale condition and coercivity, Rendiconti del Cir. Matem. Palermo XLIX, 521–526.
20. Kyritsi, S. and Papageorgiou, N.S. (2001), Hemivariational inequalities with the potential crossing the first eigenvalue, Bulletin Australian Mathematical Society 64, 381–393.
21. Kyritsi, S. and Papageorgiou, N.S. Nonsmooth critical point theory on closed convex sets and nonlinear hemivariational inequalities, Nonlinear Analyasis 61(2005), 373–403.
22. Lebourg, G. (1975), Valeur moyenne pour gradient généralisé, *CRAS, t.* 281, 792–795.
23. Liu, J.Q. and Li, S. (1984), Existence theorems of multiple critical points and their applications, *Kexue Tongbao* 17, 1025–1027.
24. Lieberman, G. (1988), Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Analysis* 12, 1203–1219.
25. Lindqvist, P. (1990), On the equation $div(\|Dx\|^{p-2}Dx) + \lambda|x|^{p-2}x = 0$, *Proceeding AMS* 109, 157–164.
26. Mawhin, J. and Willem, M. (1989), *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York.
27. Motreanu, D. and Radulescu, V. (2000), Existence results for inequality problems with lack of convexity, *Numerical Functional Analysis Optimization* 21, 869–884.
28. Naniewicz, Z. and Panagiotopoulos, P. (1995), *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York.
29. Radulescu, V. and Panagiotopoulos, P. (1998), Perturbations of hemivariational inequalities with constraints and applications, *Journal of Global Optimization* 12, 285–297.
30. Struwe, M. (1990), *Variational Methods*, Springer-Verlag, Berlin.
31. Szulkin, A. (1986), Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Analyse non Linéaire* 3, 77–109.

32. Tang, C.L. (1998), Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, *Proceedings AMS* 126 , 3263–3270.
33. Zhong, C.K. (1997), On Ekeland's variational proncipie and a minimax theorem, *Journal of Mathematical Analysis and Applications* 205, 239–250.